Effects of lateral walls in thermal convection

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The convective motion of a fluid in a container heated from below is considered. Exact solutions of the linearized Boussinesq equations are found when the container is either a circular dish or a rectangular channel, and when the horizontal boundaries are free boundaries. Solutions which are weakly unstable when the Rayleigh number has a value near its critical value, are discussed in some details. These eigensolutions are also used to construct non linear solutions. Special attention is paid to the axisymmetric solution in a cylindrical container. The amplitude and wave length of these circular rolls are determined, and the stability of them is discussed.

The circular rolls are found to be stable when the Rayleigh number is less than a certain supercritical value, depending on the diameter of the cylinder.

1. Introduction.

In most investigations on the convective motion in a fluid layer heated from below, the fluid layer is assumed to be of infinite horizontal extent. In many respects the solutions thus obtained, have the characteristic features of the convective motion in a container of finite size. Among the well known results for the model of infinite extent, we mention that the linearized equations determine the critical Rayleigh number for which the convective motion is set up, and also determine the size of the convection cells which are formed. The form of the cells can not be predicted from the linearized equations. However, a stability analysis of the non linear solutions gives the ranges of the Rayleigh number R above its critical value R_c, where the different cell pattern (hexagons, rectangles or rolls) may occur. This was investigated in several papers, notably by Schlüter, Lortz & Busse (1965) and by Palm, Ellingsen & Gjevik (1967).

It was found from experiments, however, that the cell patterns, under certain conditions strongly depend on the form of the container in which the motion takes place. Some attention was paid to such phenomena in papers by Koschmieder (1966,1967), Hoard, Robertson & Acrivos (1970) and by Sommerscales & Dougherty (1970). The most characteristic features in these respects are (i) in a rectangular dish, the rolls are most likely to develop with their axis parallel to the short side of the dish (reported in the first mentioned of the papers cited above), and (ii) in a circular dish, there is a tendency to a formation of concentric circular cells when R exceeds R_c. When R is further increased, these circular cells may persist up to a large value of R. They may also break up and develop into other cell forms, for instance hexagons. This was reported both by

Koschmieder and by Sommerscales & Dougherty in the papers cited above. The formation of hexagons, however, may occur without developing from the circular cells. Since the circular cells are typical wall dominated cell forms, it is not surprising that hexagons become more pronounced as the fluid depth decreases. Other effects which tend to affect the formation of circular cells are the boundary conditions at the horizontal boundaries and the degree of temperature dependence of the fluid properties. Some attention was paid to such effect by Hoard et al.

Theoretical investigations of the influence of the vertical walls were undertaken by several authors. Most relevant to the present work are the papers of Davis (1967,1968), Segel (1969) and Davies-Jones (1970). Both Davis and Segel considered a set of rolls in a rectangular container. While Davis determined stationary solutions of the linear and non linear equations applying a Galerkin procedure, the work of Segel is based on the idea that an amplitude modulation of a roll solution will be sufficient to form a solution satisfying the boundary conditions. Furthermore, a stability analysis of a non linear solution turns out to fit into this scheme. In the work of Davies-Jones the problem is attacked in another way. It is known that when the horizontal boundaries are considered as free boundaries, exact solutions of the equations can be found in some cases. Davies-Jones shows this to be the case for an infinite channel, and the critical Rayleigh number is calculated for different aspect ratios (ratio of channel width to depth), and for different wave numbers along the channel. A similar separation of variables was used by Müller (1966) for the problem of two dimensional convection in a channel with a given temperature difference between the vertical walls. It was also pointed out by Joseph (1971) that the separation can be done when axisymmetric motion in a circular dish is considered. Solutions of this type was also used by Zierep (1958,1959,1963) and by Müller (1965) for convective motion possessing rotational symmetry. However, they did not use the solutions in such a way that the effects of the lateral walls were properly incorporated.

In the present paper the possibility of solving the equations by separation of the variables is discussed further. The eigensolutions of the linearized equations are discussed in some details and the growth rates σ are found in terms of R and the horizontal scale parameter. It is found that for large containers, the exact solutions may be approximated by "twin solutions", i.e. the sum of two roll solutions with near equal wave numbers and amplitudes. Such approximations are utilized when non linear solutions are discussed, these solutions being determined by a series expansion of the eigensolutions of the linearized equations. Of special interest are the axisymmetric solutions in a circular dish. This motion is interesting not only because concentric circular cells are so pronounced in the experiments, but also because the non linear axisymmetric solution does not exist when the fluid layer is of infinite horizontal extent. True enough, it was pointed out already by Rayleigh (1916) that a Bessel function $J_o(ar)$ are as good a solution as $\cos ax$ when the linearized problem is treated. It turns out, however, that when the first order solution is proportional to $J_o(ar)$ and non linear terms are used to determine the amplitude, the integrals which are usually denoted the Landau coefficients in the amplitude equations fail to exist. This was pointed out in a report by the author (Ellingsen, 1971), where it was also shown that the same difficulty arises in the more general case where the first order solution is assumed to be a continuous spectrum of rolls, rather than a sum of rolls. This example clearly shows that there are cases where the solutions in a bounded fluid layer can not be constructed by an amplitude modulation of the solutions for an unbounded layer, or by a multiple scale analysis.

A numerical study of the non linear solutions and their stability will be given in another paper. In the present investigation the solutions are therefore discussed in some details only when R is close to R_c. However, some typical properties of the solutions are present in this approximation. It is found that (i) the non linear circular rolls have a well defined wave number q_o, and (ii) the wave number of a perturbation given to this solution can not be arbitrarily close to q_o. Both of these results are in contrast to what is found for straight rolls in an unbounded layer.

We have also been able to discuss the stability of non linear circular rolls against perturbations which are not axisymmetric. The value of R for which the rolls break down is determined by a rough estimate only. However, the range of values of R for which rolls are stable is found to decrease as the diameter of the container increases.

2. Basic equations and boundary conditions.

The fluid layer under consideration is bounded by two horizontal planes a distance H apart, and by vertical walls to be specified below. With $\vec{u}$ denoting the velocity vector, $\vec{k}$ a vertical unit vector, and θ and p the deviations of temperature and pressure from those of the purely conducting (motionless) case, the momentum equation, the heat equation and the equation of continuity can be written

$$\nabla^2\vec{u} + R^{\frac{1}{2}}\vec{k}\theta = \nabla p + P^{-1}(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}), \tag{2.1}$$

$$\nabla^2\theta + R^{\frac{1}{2}}\vec{k}\cdot\vec{u} = \theta_t + \vec{u}\cdot\nabla\theta \quad , \tag{2.2}$$

$$\nabla\cdot\vec{u} = 0. \tag{2.3}$$

Here the Boussinesq approximations are used, and the equations are written in dimensionless form with the scaling length, time, velocity, temperature and pressure chosen as H, $H^2\kappa^{-1}$, $H^{-1}\kappa$, $\Delta T R^{-\frac{1}{2}}$ and $H^{-2}\kappa\nu\rho_0$, respectively. κ is the thermal diffusivity, ν the kinematic viscosity, ΔT the temperature difference between the lower and the upper boundary and ρ_0 is a standard density. The density ρ is assumed to be a linear function of the temperature T with the coefficient of expansion α, $\alpha = -\rho_0{}^{-1}d\rho/\alpha T$. R is the Rayleigh number and P the Prandtl number defined by

$$R = \frac{\alpha g H^3 \Delta T}{\kappa\nu} \quad , \qquad P = \frac{\nu}{\kappa} . \tag{2.4}$$

Considering now the boundary conditions, we shall assume that the horizontal boundaries are free boundaries held at constant temperature. This assumption is necessary to obtain the solutions in a tractable form when the lateral walls are taken into account.

But this restriction is not thought to be severe since it is known from many investigations that the solutions for different boundary conditions are qualitatively similar. The simple model we are considering is therefore thought to show the essential effects of the lateral boundaries.

With $\vec{\tau}$ denoting the viscous stress vector, we therefore write the conditions

$$(2.5) \qquad \vec{k} \times \vec{\tau} = 0 , \qquad w = 0 , \qquad \theta = 0 ,$$

at the horizontal boundaries. The vertical walls are assumed to be rigid and perfectly conducting walls, the temperature of which is kept with the same linear decrease with height as in the purely heat conducting case. Accordingly we can write

$$(2.6) \qquad \vec{u} = 0 , \qquad \theta = 0 ,$$

at the vertical boundaries.

We shall find it convenient to rewrite (2.1) in the following way, Letting w and ζ denote the vertical components of velocity and vorticity

$$(2.7) \qquad w = \vec{k} \cdot \vec{u} , \qquad \zeta = \vec{k} \cdot \nabla \times \vec{u} ,$$

we can write

$$(2.8) \qquad \nabla^4 w + R^{\frac{1}{2}} \nabla_1^2 \theta = P^{-1} (\nabla^2 w_t + \nabla^2 (\vec{u} \cdot \nabla w) - \nabla \cdot (\vec{u} \cdot \nabla \vec{u})_z) ,$$

$$(2.9) \qquad \nabla^2 \zeta = P^{-1} (\zeta_t + \vec{u} \cdot \nabla \zeta + (\nabla \times \vec{u}) \cdot \nabla w) .$$

Here z is the vertical coordinate and ∇_1^2 is the two-dimensional Laplacian $\nabla_1^2 = \nabla^2 - \partial^2/\partial z^2$. In the first part of the paper we shall be concerned with the linearized equations. This is partly because we want to investigate how the vertical walls affect the onset of convection, and partly because the spectrum of eigen-solutions of the linearized equations will be used to construct the non linear solution when the Rayleigh number exceeds its critical value. The equations we are going to discuss are therefore

$$(2.10) \qquad \nabla^4 w + R^{\frac{1}{2}} \nabla_1{}^2 \theta = P^{-1} \nabla^2 w_t \ ,$$

$$(2.11) \qquad \nabla^2 \theta + R^{\frac{1}{2}} w = \theta_t \ ,$$

$$(2.12) \qquad \nabla^2 \zeta = P^{-1} \zeta_t \ ,$$

together with (2.3) and the boundary conditions (2.5) and (2.6). The linearized version of (2.1), (2.2) and (2.3) together with the boundary condition defined above constitute a self-adjoint eigenvalue problem for the time factor, as shown by Schlüter et al. (1965). It is therefore given that all the eigenvalues are real and that the critical Rayleigh number is associated with a steady solution.

3. Linear solutions for a circular dish.

In this section we consider the convective motion of a fluid in a circular dish with depth H and diameter D, such that the dimensionless radius a is

$$a = D/2H. \tag{3.1}$$

A coordinate system is chosen such that the free boundaries are located at $z = 0$ and $z = 1$, and the rigid walls at $r = a$, in terms of the cylindrical coordinates (r,ϕ,z). The velocity vector is

$$\vec{u} = \vec{i}_r u + \vec{i}_\phi v + \vec{k} w , \tag{3.2}$$

and the boundary coordinates discussed above may be written

$$u_z + w_r = r^{-1} w_\phi + v_z = w = \theta = 0 \tag{3.3}$$

at $z = 0$ and $z = 1$, and

$$u = v = w = \theta = 0 , \quad \text{at} \quad r = a. \tag{3.4}$$

In addition to these conditions we require the solutions to be regular at $r = 0$. Seeking solutions separable in r,ϕ and z, we write :

$$\theta = \exp(\sigma t) \sin \pi z \cos n\phi \, \theta(r) , \tag{3.5}$$

$$w = \exp(\sigma t) \sin \pi z \cos n\phi \, w(r), \tag{3.6}$$

$$\zeta = \exp(\sigma t) \cos \pi z \sin n\phi \, \zeta(r). \tag{3.7}$$

The functions $w(r)$, $\theta(r)$ and $\zeta(r)$ are determined from (2.10), (2,11) and(2.12). With these functions known, the horizontal velocity components u and v can be found from the equation of continuity and the definition of ζ,

$$(3.8) \qquad (ru)_r + v_\phi = - rw_z \, ,$$

$$(3.9) \qquad (rv)_r - u_\phi = r\zeta .$$

The solutions obtained in this way are easily seen to satisfy the boundary conditions at $z = 0$ and $z = 1$. The boundary conditions at $r = a$ will then yield the characteristic equation which determines the eigenvalues σ_k $(k = 1,2,\cdots)$. It is also worth noticing that the higher modes in the z-coordinate are determined in the same way by replacing $\sin\pi z$ and $\cos\pi z$ above by $\sin m\pi z$ and $\cos m\pi z$ $(m = 2,3,\cdots)$. We are thus able to derive a three parameter family of eigen-solutions with eigenvalues σ_{mnk} for a given Rayleigh number.

When (3.5) and (3.6) are introduced into (2.10) and (2.11), the functions $w(r)$ and $\theta(r)$ are found in terms of Bessel functions. We shall find it convenient to put the solutions in the following form

$$(3.10) \qquad \theta(r) = \sum_{i=1}^{3} Q_i \frac{J_n(q_i r)}{J_n(q_i a)} \quad ,$$

$$(3.11) \qquad w(r) = R^{-\frac{1}{2}} \sum_{i=1}^{3} (q_i{}^2 + \pi^2 + \sigma) Q_i \frac{J_n(q_i r)}{J_n(q_i a)} \quad .$$

q_1, q_2, q_3 are solutions of the bicubic equation

$$(3.12) \qquad (q^2 + \pi^2 + \sigma)(q^2 + \pi^2 + \sigma P^{-1})(q^2 + \pi^2) - q^2 R = 0 \; ,$$

and when Q_1, Q_2 and Q_3 are chosen as

$$(3.13) \qquad Q_1 = q_2^2 - q_3^2, \quad Q_2 = q_3^2 - q_1^2, \quad Q_3 = q_1^2 - q_2^2$$

we see that $\theta(a) = w(a) = 0$.

According to (3.7) and (2.12), the vorticity distribution $\zeta(r)$ may be written

$$(3.14) \qquad \zeta(r) = Q_4 \frac{J_n(q_4 r)}{J_n(q_4 a)}, \qquad q_4^2 + \pi^2 + \sigma P^{-1} = 0.$$

By means of (3.8) and (3.9), together with the separation

$$(3.15) \qquad u = \exp(\sigma t) \cos \pi z \cos n\phi \, u(r),$$

$$(3.16) \qquad v = \exp(\sigma t) \cos \pi z \sin n\phi \, v(r),$$

we obtain

$$(3.17) \qquad u(r) = \pi R^{-\frac{1}{2}} \sum_{i=1}^{3} (q_i^2+\pi^2+\sigma) Q_i \frac{J_n'(q_i r)}{q_i J_n(q_i a)} + \frac{nQ_4}{q_4^2} \frac{J_n(q_4 r)}{r J_n(q_4 a)},$$

$$(3.18) \qquad v(r) = -n\pi R^{-\frac{1}{2}} \sum_{i=1}^{3} (q_i^2+\pi^2+\sigma) Q_i \frac{J_n(q_i r)}{q_i^2 r J_n(q_i a)} - Q_4 \frac{J_n'(q_4 r)}{q_4 J_n(q_4 a)}.$$

From the boundary conditions $u(a) = v(a) = 0$ we then obtain the amplitude of the vorticity

$$(3.19) \qquad Q_4 = -\frac{n\pi R^{-\frac{1}{2}}(\pi^2+\sigma)q_4}{a} \frac{J_n(q_4 a)}{J_n'(q_4 a)} \sum_{i=1}^{3} q_i^{-2} Q_i,$$

and the characteristic equation

$$(3.20) \qquad \sum_{i=1}^{3} (q_i^2+\pi^2+\sigma) Q_i \frac{J_{n+1}(q_i a)}{q_i J_n(q_i a)} + \frac{n(\pi^2+\sigma)}{a} \frac{J_{n+1}(q_4 a)}{J_n'(q_4 a)} \sum_{i=1}^{3} q_i^{-2} Q_i = 0.$$

Since q_1, q_2 and q_3 are given functions of R and σ through (3.12), (3.20) determines the growth rate σ for the various modes as a function of R and a. By putting $\sigma = 0$, we also obtain the values of R for marginal instabilities as a function of a.

4. Approximate solutions for a large circular dish.

When the container in which the motion takes place is of large horizontal extent compared to the depth of the layer, the critical Rayleigh number R_c will exceed that for an unbounded layer by a small amount. We shall in this section discuss the solutions for such large containers, considering only that part of the spectrum which have a slow growth rate when R is given a value near its critical value. Since the critical Rayleigh number for the unbounded layer is $27\pi^4/4$, we write

$$(4.1) \qquad R = \frac{27\pi^4}{4}(1+\eta) .$$

The smallness of σ will be relevant expressed through the parameter

$$(4.2) \qquad \hat{\sigma} = \frac{\sigma}{\pi^2}(1+P^{-1}) ,$$

and we shall assume η and $\hat{\sigma}$ to be small compared to unity, both of them being of the same order. This assumption holds both for large and small Prandtl numbers. This only expresses the fact that the time scale used in (2.1) and (2.2) is not the proper one for all P, and (4.2) indicates the correct characteristic time to be $H^2(\kappa^{-1}+\nu^{-1})$ rather than $H^2\kappa^{-1}$.

It will also be seen below that the growth rate referred to this time scale will be virtually independent of P.

The approximate solution of (3.12) are found to be

$$(4.3) \qquad q_1{}^2 = \frac{\pi^2}{2}(1+(3\eta-2\hat{\sigma})^{\frac{1}{2}}) + O(\eta,\hat{\sigma}) ,$$

$$(4.4)\qquad q_2{}^2 = \frac{\pi^2}{2}(1 - (3\eta - 2\hat{\sigma})^{\frac{1}{2}}) + O(\eta,\hat{\sigma}) ,$$

$$(4.5)\qquad q_3{}^2 = - 4\pi^2 + O(\eta,\hat{\sigma}) .$$

We are at this stage able to draw some conclusions about the motion and the temperature distribution in the fluid.

First we note that q_3 and q_4 are imaginary. $J_n(q_3r)$ and $J_n(q_4r)$ are therefore exponential rather than oscillatory. Since the asymptotic expansion of the modified Bessel function $I_n(z) = i^{-n}J_n(iz)$ has the leading term

$$(4.6)\qquad I_n(z) \sim (2\pi)^{-\frac{1}{2}}(n^2+z^2)^{-\frac{1}{4}}\exp\{(n^2+z^2)^{\frac{1}{2}} - n\sinh^{-1}(\tfrac{n}{z})\},$$

we find for $(a-r)/a << 1$ and $a >> 1$

$$(4.7)\qquad \frac{J_n(q_3r)}{J_n(q_3a)} = \exp\{-2\pi(a-r)(1 + (\tfrac{n}{2\pi a})^2)^{\frac{1}{2}}\} ,$$

$$(4.8)\qquad \frac{J_n(q_4r)}{J_n(q_4a)} = \exp\{-\pi(a-r)(1 + (\tfrac{n}{\pi a})^2)^{\frac{1}{2}}\} .$$

These expansions are valid both for small and large azimuthal wave numbers n, provided $\hat{\sigma}$ is small for that value of n when η is small. The terms proportional to Q_3 and Q_4 in (3.10), (3.11), (3.17) and (3.18) are therefore virtually zero outside a thin layer near the outer wall. There are in fact two such layers, determined from the Q_3 and Q_4 terms of the solution. When the thicknesses δ_3 and δ_4 are defined as the distance through which the amplitudes are decreased by a factor $\exp(-\pi)$, we find from (4.7) and (4.8)

$$(4.9)\qquad \delta_3 = \tfrac{1}{2}(1 + (\tfrac{n}{2\pi a})^2)^{-\frac{1}{2}} ,$$

(4.10) $\delta_4 = (1 + (\frac{n}{\pi a})^2)^{-\frac{1}{2}}$

The layers are therefore approximately as thick as the depth of the fluid, and are decreasing with increasing n. The latter of these layers is connected with the horizontal circulation of the fluid, and vanishes when the motion becomes axisymmetric ($n = 0$).

It should be noted that these layers are not viscous or thermal boundary layers. They do not define regions where viscosity and thermal diffusivity are more important than in other regions, and their thicknesses are determined from the geometry of the container rather than the values of ν and κ.

Another interesting fact which is easily seen, is that the amplitudes of these wall layer solutions are small compared to the oscillating part of the solutions. This is because $\Sigma q_i^{-2} Q_i$ is proportional to $q_1^2 - q_2^2$, and therefore both Q_3 and Q_4 are small of order $(3\eta - 2\hat{\sigma})^{\frac{1}{2}}$.

The most pronounced effect of the vertical walls is therefore not these wall layers, but rather the existence of a "twin solution", i.e. a sum of two roll solutions with nearly equal wave numbers. As will be seen below, these two roll solutions also have nearly equal amplitudes.

By means of (4.3), (4.4) and (4.5), the characteristic equation (3.20) may be written

$$\frac{J_{n+1}(q_1 a)}{J_n(q_1 a)} - \frac{J_{n+1}(q_2 a)}{J_n(q_2 a)} \tag{4.11}$$

$$- \frac{5(q_1 - q_2)}{9\sqrt{2}\pi} \left\{\frac{J_{n+1}(q_1 a)}{J_n(q_1 a)} + \frac{J_{n+1}(q_2 a)}{J_n(q_2 a)} + \frac{2\sqrt{2}}{5}\right\} = O(\eta, \hat{\sigma}) .$$

Using the asymptotic expansions of the Bessel functions, we find

(4.12) $\quad q_1 a - q_2 a = k\pi \,, \qquad k = 1,2,3,\cdots.$

to the first approximation. To the next approximation (4.12) will be replaced by

(4.13) $\quad q_1 a - q_2 a = k\pi(1 + \frac{\lambda}{a}) \,,$

where λ is a constant to be determined from (5.11). It turns out to be

(4.14) $\quad \lambda = \frac{1}{9\pi}\{1 + (-1)^{n+k}(\sin\sqrt{2}\pi a - \frac{5}{\sqrt{2}} \cos\sqrt{2}\pi a)\} \,.$

Combining with (4.3) and (4.4) the following expression for the k'th eigenvalue is obtained,

(4.15) $\quad \hat{\sigma}_k = \frac{3}{2}\,\eta - \frac{k^2}{a^2}(1 + 2\,\frac{\lambda}{a}) \,,$

and the exchange of stabilities for the k'th mode occurs for the Rayleigh number determined from

(4.16) $\quad \eta = \frac{2k^2}{3a^2}(1 + 2\,\frac{\lambda}{a}) \,.$

It is interesting to note that λ can be either positive or negative, and that the sign of λ changes as n varies. We are therefore in general not able to predict the azimuthal wave number of the eigensolution which first begin to grow. The critical Rayleigh number is

(4.17) $\quad R_c = \frac{27\pi^4}{4}\,(1 + \frac{8}{3}\,(\frac{H}{D})^2) + O((\frac{H}{D})^3) \,,$

and in this approximation a number of solutions become unstable at the same value of R. It must be remembered, however, that asymptotic expansions of the Bessel functions of small order are used here. The results therefore need modifications when n is not small compared to $\pi a/\sqrt{2}$.

To examine what solution will be realized in an experiment, the non linear terms must be taken into account. In this respect the situation is similar to that of an infinite layer. There is an important difference, however. Consider an axisymmetric solution $n = 0$. If the diameter of the dish is 30 times the fluid depth, giving $a = 15$, there are not more than 5 unstable modes when R is 10 per cent above its critical value. If $a = 20$, the number of unstable modes are 7 at the same value of R. These results are quite different from what is found when a is taken to be infinite. In that case there are an infinite number of unstable solutions proportional to $J_0(qr)$ where q is any number between $0.8\pi/\sqrt{2}$ and $1.3\pi/\sqrt{2}$, approximately.

In Table 1 and Table 2 are given some exact solutions of the characteristic equation. The tables clearly show what is mentioned above, that the solutions are virtually independent of both the Prandtl number and the azimuthal wave number, at least for n up to 10. The variation of the eigenvalues $\hat{\sigma}$ with η are shown in Fig. 1. Also these solutions are calculated for $a = 15$. We point out the important result that the asymptotic solutions are good approximations for a container of that size (and larger).

To discuss the form of the eigensolutions, let us consider the temperature distribution $\theta_k(r)$. When the wall layer solutions are neglected, it can be written

$$(4.18) \qquad \theta_k(r) = C_k\{J_n(\frac{\pi r}{\sqrt{2}} + \frac{k\pi r}{2a}) - (-1)^k J_n(\frac{\pi r}{\sqrt{2}} - \frac{k\pi r}{2a})\} .$$

The solutions will be some different as k is odd or even. When k is odd, the amplitude near the center is of order $a^{\frac{1}{2}}$ compared to the amplitude near the outer wall. When k is even, the approximation

$$(4.19) \qquad \theta_k(r) = 2C_k \sin \frac{k\pi r}{2a} J_n'(\frac{\pi r}{\sqrt{2}})$$

will be valid for all r provided k/a is small. The motion is depressed near the center and has a sinusoidal amplitude modulation. It is interesting to realize the well-defined wave number which, in the present approximation, equals $\pi/\sqrt{2}$. In general this wave number q_o is $\frac{1}{2}(q_1+q_2)$, and since q_1 and q_2 are close to another, q_o will be approximately equal to the value of q for which (3.12) has a double zero for a given R. When σ^2 is neglected, we find this solution to be $q_o = (R/27)^{\frac{1}{4}}$. For $n = 0$, the interpretation is that the wave length of axisymmetric circular cells tend to decrease for increasing R. This will be discussed later.

	mode 1	mode 2	mode 3	mode 4	mode 5
n = 0	0.0030	0.0120	0.0270	0.0480	0.0771
n = 1	0 0030	0.0116	0.0270	0.0475	0.0750
n = 2	0.0029	0.0120	0.0259	0.0481	0.0741
n = 3	0.0030	0.0113	0.0271	0.0460	0.0747
n = 5	0.0030	0.0111	0.0272	0.0457	0.0738
n = 10	0.0032	0.0111	0.0295	0.0446	0.0813

Table 1

Values of η for which exchange of stabilities occur for different modes and different azimuthal wave numbers n. Radius a = 15.

		mode 1	mode 2	mode 3	mode 4	mode 5
P = ∞	n = 0	0.0694	0.0785	0.0946	0.1153	0.1462
	n = 10	0 0697	0 0778	0 0961	0 1150	0 1452
P = 1	n = 0	0.0705	0.0797	0.0957	0.1164	0.1472
	n = 10	0.0707	0.0789	0.0972	0.1159	0.1460
P = 0	n = 0	0.0695	0.0785	0.0946	0.1154	0.1461
	n = 10	0.0697	0.0777	0.0961	0.1144	0.1449

Table 2

Values of η for a given growth rate $\hat{\sigma} = 0.10$ and for different Prandtl numbers. Radius a = 15.

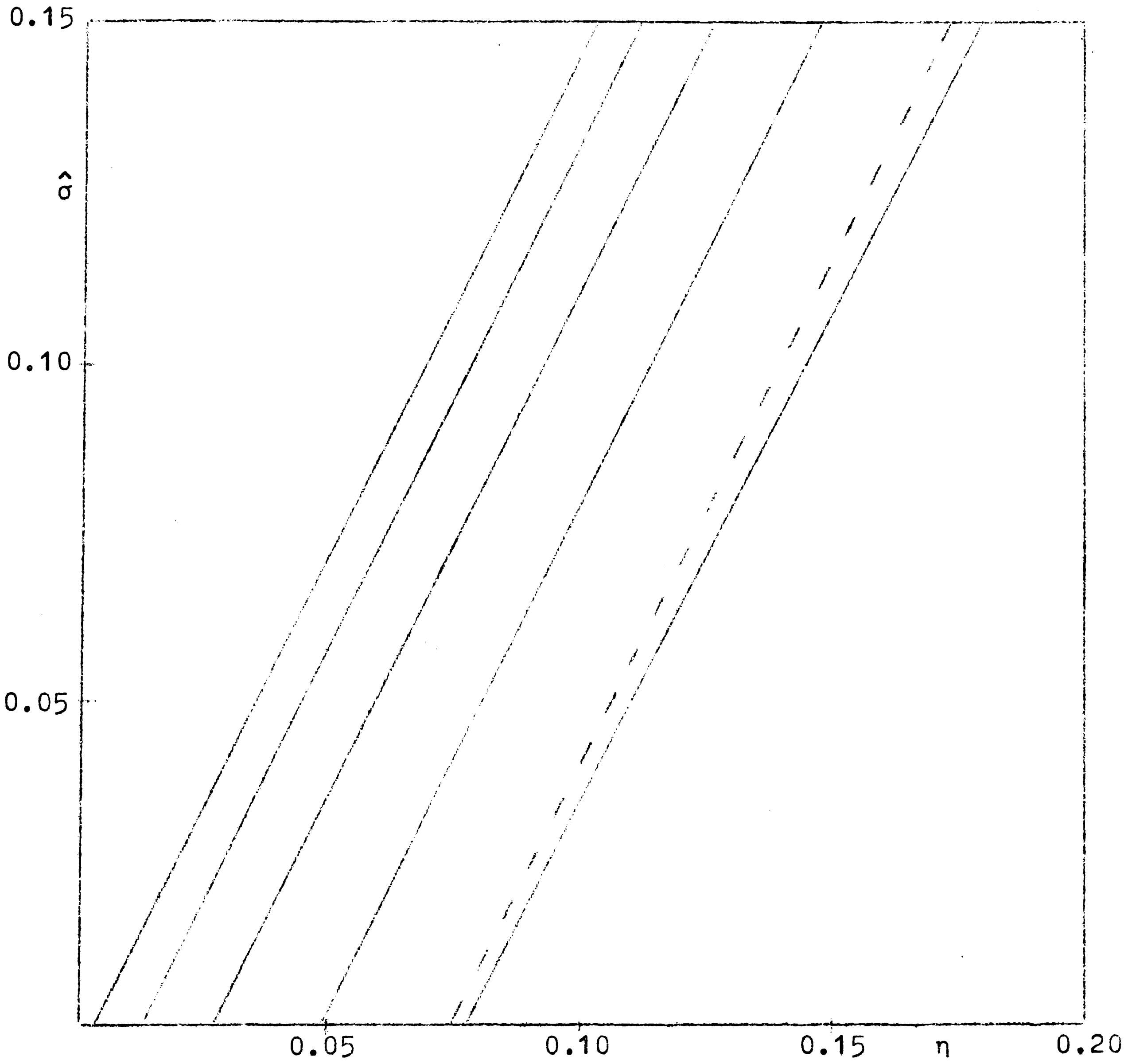

Fig. 1.

Exact solutions for a circular dish of radius $a = 15$, mode $1 - 5$.
Dashed line: asymptotic solution for the 5. mode,
$\hat{\sigma} = \frac{\sigma}{\pi^2}(1 + P^{-1})$, $R = \frac{27\pi^4}{4}(1 + \eta)$.

5. Linear solutions for an infinite channel.

Rectangular containers are frequently used in experiments on convection. The present method of obtaining exact solutions by separation of the variables is not applicable in this case, even if the horizontal boundaries are considered as free boundaries. If, however, the length of the rectangle is so large, compared to the width, that the effect of the distant walls may be neglected and the container can be treated as an infinite channel, some exact solutions are given by Davies-Jones [1970] and discussed in some details when the solutions are steady, having a sinusoidal variation along the channel. As we shall be interested in the spectrum of weakly unstable eigensolutions when R is near its critical value, we shall shortly discuss the solutions of (2.10) to (2.12) with the boundary conditions (2.5) and (2.6). Special attention will be paid to the asymptotic solutions when the channel width is large compared to the depth of the fluid layer.

The channel is taken to be of depth H and width B. A cartesian coordinate system (x,y,z) is applied to the channel such that the horizontal boundaries coincide with the planes $z = 0$ and $z = 1$, and the vertical walls coincide with the planes $x = \pm b$. With the scaling defined in section 2, b is then given by

$$b = B/2H \, . \tag{5.1}$$

When the motion is assumed to be sinusoidal with wavenumber κ along the channel, the solutions for w, θ, ζ may be separated in the following way.

$$\theta = \exp(\sigma t)\sin\pi z \, \cos\kappa y \; \theta(x), \tag{5.2}$$

$$w = \exp(\sigma t)\sin\pi z \, \cos\kappa y \; w(x), \tag{5.3}$$

$$\zeta = \exp(\sigma t)\cos\pi z \, \sin\kappa y \; \zeta(x). \tag{5.4}$$

There are two cases to be treated separately, the symmetric case in which $\theta(-x) = \theta(x)$ and the antisymmetric one with $\theta(-x) = -\theta(x)$. In both cases the solutions $\theta(x)$, $w(x)$ and $\zeta(x)$ can be given a form analogous to the solutions in section 3. In fact, the solutions are obtained if the substitutions

$$(5.5)\qquad J_n(q_i r)\Big/J_n(q_i a) \to \cos p_i x\Big/\cos p_i b \quad ,$$

$$(5.6)\qquad J_n(q_i r)\Big/J_n(q_i a) \to \sin p_i x\Big/\sin p_i b \ ,$$

are introduced in (3.10), (3.11) and (3.14). p_i is defined by

$$(5.7)\qquad p_i^2 = q_i^2 - \kappa^2.$$

The horizontal velocities are then found, and by using the boundary conditions, we find the characteristical equations in the form

$$(5.8)\quad \sum_{i=1}^{3}(q_i^2+\pi^2+\sigma)Q_i q_i^{-2}p_i \tan p_i b+\kappa^2(\pi^2+\sigma)p_4^{-1}\tan p_4 b\sum_{i=1}^{3}q_i^{-2}Q_i = 0$$

for the symmetric case, and

$$(5.9)\quad \sum_{i=1}^{3}(q_i^2+\pi^2+\sigma)Q_i q_i^{-2}p_i \cot p_i b+\kappa^2(\pi^2+\sigma)p_4^{-1}\cot q_4 b\sum_{i=1}^{3}q_i^{-2}Q_i = 0 \ ,$$

for the antisymmetric case.

When the expression for the vorticity amplitude Q_4 are derived, they are found to be proportional to the sum $\sum_{i=1}^{3} q_i^{-2}Q_i$ in both cases.

In the discussion of the equations (5.8) and (5.9) we first note that since q_i and Q_i have the same meaning as in section 3, both p_3 and p_4 will be imaginary according to (5.7), and approximately

(5.10) $$|p_3| = 2\pi(1 + \frac{\kappa^2}{4\pi^2})^{\frac{1}{2}} ,$$

(5.11) $$|p_4| = \pi(1 + \frac{\kappa^2}{\pi^2})^{\frac{1}{2}} .$$

The corresponding terms in the temperature and velocity distributions therefore represent wall layers of the same type as those discussed in section 4. The thicknesses are

(5.12) $$\delta_3 = \tfrac{1}{2}(1 + \frac{\kappa^2}{4\pi^2})^{-\frac{1}{2}}$$

(5.13) $$\delta_4 = (1 + \frac{\kappa^2}{\pi^2})^{-\frac{1}{2}} ,$$

decreasing with increasing κ. The amplitudes of these solutions are proportional to Q_3 and Q_4 which are proportional to $q_1^2 - q_2^2$ and therefore small.

When b is large compared to one, the dominating terms in the characteristic equations (5.8) and (5.9) give

(5.14) $$p_1 b \tan p_1 b - p_2 b \tan p_2 b = 0 ,$$

(5.15) $$p_1 b \cot p_1 b - p_2 b \cot p_2 b = 0 .$$

If $\pi^2/2 - \kappa^2$ is positive and large compared to $q_1^2 - \pi^2/2$, both (5.14) and (5.15) give

(5.16) $$p_1 b - p_2 b = k\pi , \qquad k = 1,2,3 \cdots$$

Combining with (4.3) and (4.4) we find

(5.17) $$\hat{\sigma} = \frac{3}{2}\eta - \frac{2k^2}{\pi^2 b^2}(\frac{\pi^2}{2} - \kappa^2)$$

for the k'th node.

The horizontal temperature distribution $\theta_k(x,y)$ can be written

(5.18) $$\theta_k(x,y) = C_k \cos\frac{k\pi x}{2b}\cos\left(\frac{\pi^2}{2}-\kappa^2\right)^{\frac{1}{2}}x\,\cos\kappa y$$

or

(5.19) $$\theta_k(x,y) = C_k \sin\frac{k\pi x}{2b}\sin\left(\frac{\pi^2}{2}-\kappa^2\right)^{\frac{1}{2}}x\,\cos\kappa y$$

as k is odd or even, respectively. The solutions therefore define a system of rectangular rolls with over all wave number $\pi/\sqrt{2}$ and an amplitude modulation $\cos\frac{k\pi x}{2b}$ or $\sin\frac{k\pi x}{2b}$. From (5.17) it is seen that the solutions become more unstable as the cells are stretched in the x-direction. The most unstable eigensolutions are those for which κ is close to $\pi/\sqrt{2}$, i.e. solutions which are nearly straight rolls with axes perpendicular to the channel walls. To investigate these solutions we use the definition of p_1 and p_2 to obtain

(5.20) $$p_1{}^2 - p_2{}^2 = \pi^2(3\eta - 2\hat{\sigma})^{\frac{1}{2}} ,$$

(5.21) $$p_1{}^2 + p_2{}^2 = \pi^2 - 2\kappa^2 .$$

It is then found

(5.22) $$\hat{\sigma} = \frac{3\eta}{2} - \frac{1}{2\pi^4 b^4}\{(p_1 b)^2 - (p_2 b)^2\}^2 ,$$

(5.23) $$\kappa^2 = \frac{\pi^2}{2}\{1 - \frac{1}{\pi^2 b^2}((p_1 b)^2 + (p_2 b)^2)\} .$$

where $p_1 b$ and $p_2 b$ are of order 1. The results found here are in accordance with the results of Segel [1969]. The correction terms due to lateral walls perpendicular to the roll axes are of order b^{-4} while the terms due to the walls parallel to the rolls (corresponding to

$\kappa = 0$ above) are of order b^{-2}. It is also seen that when κ is close to $\pi/\sqrt{2}$, p_2 becomes imaginary. The amplitude modulation is therefore partly sinusoidal and partly exponential.

6. Non linear solutions.

From the previous sections the eigenvalues and eigensolutions for the linearized problem are considered. A three parameter family of solutions are found both for the circular dish and for the infinite channel. For each z-dependence of the form $\sin m\pi z$, $m = 1,2,\cdots$, there is a two parameter family of solutions. While only $m = 1$ is considered above, the modifications necessary to obtain the solutions for $m > 1$ are obvious. In the case of a circular dish the spectrum of eigenvalues is discrete, dependent on the three integers m,n and k. Here m is the vertical wave number, n the azimuthal wave number while k denotes the k'th mode in the radial direction. For the infinite channel the three parameters are m, κ and k, the latter now denoting the mode in the x-direction which is across the channel. κ is the wave number along the channel (the y-direction), and since the channel is assumed to be of infinite length, there are no restrictions on κ. The spectrum is therefore continuous in κ.

A solution of the non linear equations (2.1) and (2.2) will now be discussed. The method of solution will be a Galerkin procedure where the eigensolutions of the linearized equations are used as the trial functions. The problem will thus be reduced to a set of coupled equations for the time dependent coefficients in the expansion series. The dominating terms in this expansion when the Rayleigh number is near its critical value will be those eigensolutions which are linearly unstable, i.e. those eigensolutions which are discussed in the previous sections.

The solutions we are going to determine will be written in the following form

(6.1) $$\vec{u} = \vec{u}^{(1)} + \vec{u}^{(2)} + \dots ,$$

(6.2) $$\theta = \theta^{(1)} + \theta^{(2)} + \dots ,$$

where

(6.3) $$\vec{u}^{(m)} = \sum_\alpha A_\alpha^{(m)}(t)\ \vec{u}_\alpha^{(m)}(\vec{r}) ,$$

(6.4) $$\theta^{(m)} = \sum_\alpha A_\alpha^{(m)}(t)\ \theta_\alpha^{(m)}(\vec{r}).$$

Here α denotes an index-pair (n,k) or (κ,k) such that $\vec{u}_\alpha^{(m)}$ and $\theta_\alpha^{(m)}$ are the eigensolutions (m,n,k) or (m,κ,k). The corresponding eigenvalues are $\sigma_\alpha^{(m)}$. The amplitudes $A_\alpha^{(1)}(t)$ are assumed to be small of order $\varepsilon = (\max \sigma_\alpha^{(1)})^{\frac{1}{2}}$. $\vec{u}_\alpha^{(m)}$ and $\theta_\alpha^{(m)}$ satisfy the equations

(6.5) $$\nabla^2\vec{u}_\alpha^{(m)} + R^{\frac{1}{2}}\vec{k}\ \theta_\alpha^{(m)} = \nabla p_\alpha^{(m)} + P^{-1}\ \sigma_\alpha^{(m)}\vec{u}_\alpha^{(m)},$$

(6.6) $$\nabla^2\theta_\alpha^{(m)} + R^{\frac{1}{2}}\vec{k}\cdot\vec{u}^{(m)} = \sigma_\alpha^{(m)}\ \theta_\alpha^{(m)}.$$

Due to the self-adjointness of the operator defined by the left hand side of (6.5) and (6.6), the eigensolutions are orthogonal to each other. They will also be normalized so that

(6.7) $$V^{-1}\int_V \{P^{-1}\vec{u}_\alpha^{(m)}\cdot\vec{u}_\beta^{(n)} + \theta_\alpha^{(m)}\theta_\beta^{(n)}\}dV = \delta_{mn}\delta_{\alpha\beta} ,$$

V being the fluid volume. From (2.1), (2.2), (6.5) and (6.6) the following set of equations are derived

$$(6.8)\qquad \dot{A}_\alpha^{(m)} - \sigma_\alpha^{(m)} A_\alpha^{(m)} = - V^{-1}\int_V \{P^{-1}\vec{u}_\alpha^{(m)}\cdot(\vec{u}\cdot\nabla\vec{u}) + \theta_\alpha^{(m)}\vec{u}\cdot\nabla\theta\}dV \ .$$

When terms of higher order than the third order in ε are omitted, the amplitude equations take the form

$$(6.9)\qquad \dot{A}_\alpha^{(2)} - \sigma_\alpha^{(2)} A_\alpha^{(2)} = - \sum_{\beta,\gamma} A_\beta^{(1)} A_\gamma^{(1)} M_{\alpha\beta\gamma}^{(2)} \ ,$$

$$(6.10)\qquad \dot{A}_\alpha^{(1)} - \sigma_\alpha^{(1)} A_\alpha^{(1)} = - \sum_{\beta,\gamma} A_\beta^{(1)} A_\gamma^{(2)} M_{\alpha\beta\gamma}^{(1)} \ .$$

The coefficients are defined by the integrals

$$(6.11)\qquad M_{\alpha\beta\gamma}^{(2)} = V^{-1}\int_V \{P^{-1}\vec{u}_\alpha^{(2)}\cdot(\vec{u}_\beta^{(1)}\cdot\nabla\vec{u}_\gamma^{(1)}) + \theta_\alpha^{(2)}\vec{u}_\beta^{(1)}\cdot\nabla\theta_\gamma^{(1)}\}dV \ ,$$

$$(6.12)\qquad M_{\alpha\beta\gamma}^{(1)} = V^{-1}\int_V \{P^{-1}\vec{u}_\alpha^{(1)}\cdot(\vec{u}_\beta^{(1)}\cdot\nabla\vec{u}_\gamma^{(2)} + \vec{u}_\gamma^{(2)}\cdot\nabla\vec{u}_\beta^{(1)}) + \theta_\alpha^{(1)}(\vec{u}_\beta^{(1)}\cdot\nabla\theta_\gamma^{(2)} + \vec{u}_\gamma^{(2)}\cdot\nabla\theta_\beta^{(1)})\}dV.$$

(6.9) and (6.10) are not the most convenient form of the amplitude equations. This is because a number of second order eigensolutions $A_\alpha^{(2)}$ may be excited even when only one first order eigensolution $A_\beta^{(1)}$ is present. We therefore make use of the transformation used by Eckhaus [1965] for a similar problem. Since $\sigma_\alpha^{(2)}$ is of order 1, $\dot{A}_\alpha^{(2)}$ is negligible compared to $A_\alpha^{(2)}$ and therefore the second order solutions may be written

$$\vec{u}^{(2)} = \sum_{\alpha} A_{\alpha}^{(2)} u_{\alpha}^{(2)} = \sum_{\beta\gamma} A_{\beta}^{(1)} A_{\gamma}^{(1)} \vec{v}_{\beta\gamma} , \tag{6.13}$$

$$\theta^{(2)} = \sum_{\alpha} A_{\alpha}^{(2)} \theta_{\alpha}^{(2)} = \sum_{\beta\gamma} A_{\beta}^{(1)} A_{\gamma}^{(1)} \Theta_{\beta\gamma} , \tag{6.14}$$

where $\vec{v}_{\beta\gamma}(\vec{r})$ and $\Theta_{\beta\gamma}(\vec{r})$ satisfy

$$\nabla^2 \vec{v}_{\beta\gamma} + R^{\frac{1}{2}} \vec{k}\, \Theta_{\beta\gamma} = \nabla \pi_{\beta\gamma} + P^{-1} \vec{u}_{\beta}^{(1)} \cdot \nabla \vec{u}_{\gamma}^{(1)} , \tag{6.15}$$

$$\nabla^2 \Theta_{\beta\gamma} + R^{\frac{1}{2}} \vec{k} \cdot \vec{v}_{\beta\gamma} = \vec{u}_{\beta}^{(1)} \cdot \nabla \theta_{\gamma}^{(1)} , \tag{6.16}$$

and the appropriate boundary conditions. Alternatively to (6.9) and (6.10) we have the amplitude equations

$$\mathring{A}_{\alpha}^{(1)} - \sigma_{\alpha}^{(1)} A_{\alpha}^{(1)} = - \sum_{\beta\gamma\delta} A_{\beta}^{(1)} A_{\gamma}^{(1)} A_{\delta}^{(1)} M_{\alpha\beta\gamma\delta} . \tag{6.17}$$

The coefficients are now given by

$$\begin{aligned} M_{\alpha\beta\gamma\delta} = V^{-1} \int_V \{ & P^{-1} \vec{u}_{\alpha}^{(1)} \cdot (\vec{u}_{\beta}^{(1)} \cdot \vec{v}_{\gamma\delta} + \vec{v}_{\gamma\delta} \cdot \nabla \vec{u}_{\beta}^{(1)}) \\ & + \theta_{\alpha}^{(1)} (\vec{u}_{\beta}^{(1)} \cdot \nabla \Theta_{\gamma\delta} + \vec{v}_{\gamma\delta} \cdot \nabla \theta_{\beta}^{(1)}) \} dV . \end{aligned} \tag{6.18}$$

The equivalence between the two approaches is expressed through

$$M_{\alpha\beta\gamma\delta} = \sum_{\kappa} \frac{1}{\sigma_{\kappa}^{(2)}} M_{\alpha\beta\kappa}^{(1)} M_{\kappa\gamma\delta}^{(2)} , \tag{6.19}$$

which is obtained by eliminating $A_{\alpha}^{(2)}$ from (6.9) and (6.10) and comparing with (6.17).

Since we shall be concerned with large containers (cylindrical dish with radius $a >> 1$ and channel with half width $b >> 1$), there are some approximations which can be done in the integral (6.18). The first one is to neglect the wall layer solutions discussed in sections 4 and 5. The regions where these terms are significant are of order a^{-1} or b^{-1} compared to the total volume. The vertical component of the vorticity is thus neglected in the first order solutions. It is seen, however, that the same quantity is negligible also to the second order. From (2.9) we find

$$\nabla^2(\vec{k}\cdot\nabla\times\vec{v}_{\beta\gamma}) = P^{-1}(\vec{u}^{(1)}_{\beta}\cdot\nabla\zeta^{(1)}_{\gamma} + (\nabla\times\vec{u}^{(1)}_{\beta})\cdot\nabla w^{(1)}_{\gamma}) \tag{6.20}$$

to be satisfied in addition to (6.15) and (6.16). The solutions found in section 3 and 5 make the right hand side of (6.20) proportional to the difference in the wave numbers q_1 and q_2. The oscillatory part of $\vec{k}\cdot\nabla\times\vec{v}_{\beta\gamma}$ will therefore be of order a^{-1} or b^{-1}.

The contribution from the last term in the integral (6.18) will now be shown to be small. When $\vec{k}\cdot\nabla\times\vec{v}_{\gamma\delta}$ is set equal to zero, the horizontal component of $\vec{v}_{\gamma\delta}$ is derived from a potential χ. Since $\theta^{(1)}_{\alpha}$ and $\theta^{(1)}_{\beta}$ have the same z-dependence the integral may be written

$$\int_V \theta^{(1)}_{\alpha}\vec{v}_{\gamma\delta}\cdot\nabla\theta^{(1)}_{\beta}dV = \tfrac{1}{2}\int_V \chi(\theta^{(1)}_{\beta}\nabla_1{}^2\theta^{(1)}_{\alpha} - \theta^{(1)}_{\alpha}\nabla_1{}^2\theta^{(1)}_{\beta})dV. \tag{6.21}$$

The integral is proportional to the difference in the wave numbers q_1 and q_2, and the integral is therefore small of order a^{-1} or b^{-1}. The same result will be valid also for the third term in the integral (6.18). These integrals are exactly zero when the first order solution is a sum of rolls all having the same wave number. This is the result of Schlüter et al. [1965].

In the discussion of the linear solutions the minor role of the Prandtl number was pointed out. The only noticable effect of P is just to determine the time scale on which the motion grows or decays. The role of P is more pronounced in the non linear case since the magnitude of P determines whether the convective term in the momentum equation or in the heat equation will be the dominating one. It also determine to which degree the vertical vorticity component is negligible. In what follows we will simplify the problem by considering P so large that terms proportional to P^{-1} can be omitted. The coefficients $M_{\alpha\beta\gamma\delta}$ in (6.17) are then given by

$$M_{\alpha\beta\gamma\delta} = - V^{-1}\int_V \Theta_{\gamma\delta}\ \vec{u}_\beta^{(1)}\cdot\nabla\theta_\alpha^{(1)}dV \ . \tag{6.22}$$

7. Non linear convection in a circular dish.

In section 4 it was found that in the spectrum of unstable eigensolutions many azimuthal wave numbers are present which have virtually the same critical Rayleigh number and the same growth rate. This behaviour will have a special consequence for the non linear solutions. If a single wave number n different from zero is first considered, the wave number $3n$ will be excited through the non linear coupling. If two wave numbers are initially different from zero, six new wave numbers will be excited, and so on. All of these modes are equally unstable in our approximation. Apart from the axisymmetric solution $n = 0$, there is therefore no tendency for a simple pattern to develop. The motion is most likely to be unordered and chaotic, or it may eventually develop into a pattern which is unaffected by the geometry of the container (for instance hexagons). This situation is different from the case of non linear roll solutions in an unbounded layer. In that case an unstable roll solution is not able to excite another unstable roll with initially zero amplitude, therefore one single roll is always a possible non linear solution. From such considerations, and from the experimental results referred to in the introduction, we are led to discuss an axisymmetric non linear solution. The non symmetric solutions are important, however, in that they may be able to make the symmetric solution unstable.

The amplitude equation (6.17) together with (6.22) will now be discussed in some details. It will be no confusing in neglecting the superscript in these equations, and writing A_α for $A_\alpha^{(1)}$, σ_α for $\sigma_\alpha^{(1)}$ etc. It will also be noted that the indices α,β,γ and δ originally being index-pairs of the form (n,k), are now single integer indices since $n = 0$. The eigensolution $\theta_\alpha(r)$ will now be written

$$(7.1) \qquad \theta_\alpha = \sin \pi z \; Q_\alpha \sum_i Q_{\alpha i} \frac{J_o(q_{\alpha i} r)}{J_o(q_{\alpha i} a)} \; .$$

Here $q_{\alpha i}$ $(i = 1,2,3)$ are the q-values associated with the eigenvalue σ_α, i.e. $q_{\alpha i}$ are the solutions of

$$(7.2) \qquad (q_\alpha{}^2 + \pi^2 + \sigma_\alpha)(q_\alpha{}^2 + \pi^2)^2 - q_\alpha{}^2 R = 0.$$

$Q_{\alpha i}$ depends on $q_{\alpha i}$ according to (3.13), and Q_α is a normalizing factor introduced to satisfy (6.7). For large values of a, the approximate values are found

$$\frac{Q_\alpha Q_{\alpha 1}}{J_o(q_{\alpha 1}a)} = C_{\alpha 1} = \left(\frac{\pi^2 a}{2\sqrt{2}}\right)^{\frac{1}{2}} , \tag{7.3}$$

$$\frac{Q_\alpha Q_{\alpha 2}}{J_o(q_{\alpha 2}a)} = C_{\alpha 2} = (-1)^{\alpha+1} C_{\alpha 1} . \tag{7.4}$$

The convective term may then be written

$$\vec{u}_\beta \cdot \nabla\theta_\alpha = \sin 2\pi z \; f_{\beta\alpha}(r) , \tag{7.5}$$

where

$$f_{\beta\alpha}(r) = \frac{\pi}{2} R^{-\frac{1}{2}} \sum_{ij} (q_{\beta i}{}^2 + \pi^2 + \sigma_\beta) C_{\beta i} \, C_{\alpha j} \times \left\{\frac{q_{\alpha j}}{q_{\beta i}} J_1(q_{\beta i} r) J_1(q_{\alpha j} r) + J_o(q_{\beta i} r) J_o(q_{\alpha j} r)\right\}. \tag{7.6}$$

To determine $\Theta_{\gamma\delta}$, we write

$$\Theta_{\gamma\delta} = \sin 2\pi z \; g_{\gamma\delta}(r) , \tag{7.7}$$

$$\vec{k} \cdot \vec{v}_{\gamma\delta} = \sin 2\pi z \; h_{\gamma\delta}(r) , \tag{7.8}$$

and obtain from (6.15) and (6.16) the equations

$$(\nabla_1{}^2 - 4\pi^2)^2 h_{\gamma\delta} + R^{\frac{1}{2}} \nabla_1{}^2 g_{\gamma\delta} = 0 , \tag{7.9}$$

$$(\nabla_1{}^2 - 4\pi^2) g_{\gamma\delta} + R^{\frac{1}{2}} h_{\gamma\delta} = f_{\gamma\delta}. \tag{7.10}$$

The solutions we are seeking have to satisfy the boundary conditions

$$g_{\gamma\delta}(a) = h_{\gamma\delta}(a) = 0 , \qquad \int_0^a h_{\gamma\delta}(r) r \, dr = 0 . \tag{7.11}$$

The last condition expresses the horizontal velocity to be zero.

A solution of (7.9) and (7.10) satisfying the first and second condition (7.11), but not the last one, is the Fourier-Bessel series

$$(7.12) \qquad g_{\gamma\delta}(r) = -\frac{2}{a^2} \sum_{\kappa} \frac{(p_\kappa^2+4\pi^2)^2 \bar{f}_{\gamma\delta}(p_\kappa) J_0(p_\kappa r)}{\{(p_\kappa^2+4\pi^2)^3 - p_\kappa^2 R\} J_1(p_\kappa a)^2}$$

Here $\bar{f}_{\gamma\delta}(p_\kappa)$ is the Hankel transform of $f_{\gamma\delta}(r)$,

$$(7.13) \qquad \bar{f}_{\gamma\delta}(p_\kappa) = \int_0^a f_{\gamma\delta}(r) J_0(p_\kappa r) r \, dr \ ,$$

and p_κ $(\kappa = 1,2,\dots)$ are the solutinns of

$$(7.14) \qquad J_0(p_\kappa a) = 0 \ .$$

The solution of the homogeneous equations, which have to be added to get the last condition (7.11) satisfied, has the form

$$(7.15) \qquad g_{\gamma\delta}(r) = \sum_i Q_i J_0(q_i r) \ ,$$

where q_i $(i = 1,2,3)$ satisfy

$$(7.16) \qquad (q^2 + 4\pi^2)^3 - q^2 R = 0$$

For $R = 27\pi^4/4$, the solutions are

$$(7.17) \qquad q_{1,2}^{\ 2} = -\frac{\pi^2}{8}(17 \pm 9\sqrt{3}i), \quad q_3^{\ 2} = -\frac{47\pi^2}{8} \ .$$

The complex arguments make the Bessel functions decrease exponentially from the outer wall and inwards. The solutions of the homogeneous equations are therefore wall solutions of the kind discussed in section 4. Their contributions to the integral (6.22) will be small of the same order as the terms neglected in the previous section. The approximate solution (7.12) is therefore used in (6.22). By means of the Parseval's theorem for the Hankel transform, we obtain the following expression

$$(7.18)\quad M_{\alpha\beta\gamma\delta} = \frac{2}{a^4} \sum_{\kappa} \frac{(p_\kappa{}^2+4\pi^2)^2 \bar{f}_{\beta\alpha}(p_\kappa)\bar{f}_{\gamma\delta}(p_\kappa)}{\{(p_\kappa{}^2+4\pi^2)^3 - p_\kappa{}^2 R\} J_1(p_\kappa a)^2}:$$

The asymptotic value of $M_{\alpha\beta\gamma\delta}$ for large a will now be discussed. First we note that the infinite integral

$$(7.19)\qquad \tilde{f}_{\beta\alpha}(p_\kappa) = \int_0^\infty f_{\beta\alpha}(r) J_0(p_\kappa r) r\, dr$$

can be found from known formulas (for instance Watson 1962 p. 411). The result is

$$(7.20)\quad \begin{aligned} \tilde{f}_{\beta\alpha}(p_\kappa) &= \tfrac{1}{4} R^{-\frac{1}{2}} \sum_{ij} (q_{\beta i}{}^2 + \pi^2 + \sigma_\beta) C_{\beta i} C_{\alpha j} \frac{q_{\beta i} + q_{\alpha j}}{q_{\beta i}{}^2 q_{\alpha j}} \\ &\times \left\{ \left[\frac{(q_{\beta i}+q_{\alpha j})^2 - p_\kappa{}^2}{p_\kappa{}^2 - (q_{\beta i}-q_{\alpha j})^2} \right]^{\frac{1}{2}} + \frac{q_{\beta i}-q_{\alpha j}}{q_{\beta i}+q_{\alpha j}} \left[\frac{p_\kappa{}^2 - (q_{\beta i}-q_{\alpha j})^2}{(q_{\beta i}+q_{\alpha j})^2 - p_\kappa{}^2} \right]^{\frac{1}{2}} \right\} \end{aligned}$$

when p_κ lies in the interval $(|q_{\beta i} - q_{\alpha j}|,\ q_{\beta i} + q_{\alpha j})$, outside this interval it is exactly zero. By means of (7.20) and the asymptotic expansions of the Bessel functions, an estimate of

$$(7.21)\qquad \bar{f}_{\beta\alpha}(p_\kappa) = \tilde{f}_{\beta\alpha}(p_\kappa) + \int_\infty^a f_{\beta\alpha}(r) J_0(p_\kappa r) r\, dr$$

can be found. It is here utilized that $q_{\alpha i}$ and $q_{\beta j}$ are close to $\pi/\sqrt{2}$ of order a^{-1}.

When p_n is larger than $\pi\sqrt{2}$, it is straight forward to see that each term in the sum (7.18) is of order a^{-3}. When p_κ is close to $\pi\sqrt{2}$, the integral (7.13) is transformed to Fresnel int-integrals and again the order a^{-3} is obtained. Next consider p_κ in the interval $(0,\pi\sqrt{2})$ such that p_κ and $\pi\sqrt{2}-p_\kappa$ are not small of order a^{-1}. Then $\bar{f}_{\alpha\beta} - \tilde{f}_{\alpha\beta} = O(a^{-\frac{1}{2}})$, and we find approximately

$$(7.22) \qquad \bar{f}_{\alpha\beta}(p_\kappa) = S_\alpha S_\beta \frac{2a}{\sqrt{6}} \frac{1}{p_\kappa} (2\pi^2 - p_\kappa^{\,2})^{\frac{1}{2}}$$

where $S_\alpha = \frac{1}{2}(1-(-1)^\alpha)$. Each term in the sum (7.18) is of order a^{-1}, but as a increases, the number of terms increases, and the sum will approach the integral

$$(7.23) \qquad \frac{1}{3} S_\alpha S_\beta S_\gamma S_\delta \int \frac{(p^2+4\pi^2)^2(2\pi^2-p^2)}{(p^2+4\pi^2)^3-p^2R} \frac{dp}{p} .$$

The contribution to $M_{\alpha\beta\gamma\delta}$ from p_κ in this interval will tend to a finite value for increasing a only when α,β,γ and δ are odd integers. Otherwise it will tend to zero, more rapidly as more of the integers are chosen even.

It remains to discuss the contribution from values of p_κ which are small of order a^{-1}. First we write

$$(7.24) \quad \bar{f}_{\beta\alpha}(p_\kappa) = p_\kappa^{-2}\left\{\int_0^\varepsilon f_{\beta\alpha}\left(\frac{r}{p_\kappa}\right)J_0(r)r\,dr + \int_\varepsilon^{p_\kappa a} f_{\beta\alpha}\left(\frac{r}{p_\kappa}\right)J_0(r)r\,dr\right\} ,$$

and realize that for sufficient large a, ε can be chosen so that the first integral is arbitrary small while $f_{\beta\alpha}$ can be replaced by the first term of its asymptotic expansion in the last integral. The terms we are omitting will then be of the order a^{-1}. Utilizing this, we find the contribution to $M_{\alpha\beta\gamma\delta}$ to be written in the following way

$$(7.25) \qquad \sqrt{3}\; S_{\alpha+\beta+1} S_{\gamma+\delta+1} \sum_{\kappa} \frac{(p_\kappa{}^2+4\pi^2)^2}{(p_\kappa{}^2+4\pi^2)^3-p_\kappa{}^2 R} \; \frac{I_\kappa(\alpha,\beta) I_\kappa(\gamma,\delta)}{u_\kappa{}^2 J_1(u_\kappa)^2} \;,$$

where $\quad u_\kappa = p_\kappa a \quad$ and

$$(7.26) \qquad I_\kappa(\alpha,\beta) = \int_0^{u_\kappa} J_0(u)\{\cos \frac{(\alpha-\beta)\pi u}{2u_\kappa} - (-1)^\beta \cos \frac{(\alpha+\beta)\pi u}{2u_\kappa}\}\, du \;.$$

To get $M_{\alpha\beta\gamma\delta}$ different from zero, both $\alpha+\beta$ and $\gamma+\delta$ must be even.

The sum in (7.25) is dependent on a through $p_\kappa = \frac{u_\kappa}{a}$. However, its behaviour for large a will be different as the indices are odd or even. When β is even, $I_\kappa(\alpha,\beta)$ will be of order u_κ^{-2} for large κ. When β is odd, it tends towards 2 in the limit. The consequence of this is that when all the indices α,β,γ and δ are odd integers, $M_{\alpha\beta\gamma\delta}$ will not be independent of a in the limit $a \to \infty$. Otherwise it will become a constant,

$$(7.27) \quad M_{\alpha\beta\gamma\delta} = \frac{\sqrt{3}}{4\pi^2} \sum_{\kappa} \frac{I_\kappa(\alpha,\beta) I_\kappa(\gamma,\delta)}{u_\kappa{}^2 J_1(u_\kappa)^2} \;,$$

in the limit. (7.27) is seen to correspond to using the approximate solution $g_{\gamma\delta}(r) = -\frac{1}{4\pi^2} f_{\gamma\delta}(r)$. The dependence of $M_{\alpha\beta\gamma\delta}$ on a in the former case is estimated (though not rigorously proved) to be logaritmic. Consider $M_{\alpha\beta\gamma\delta}$ to be determined by the sum (7.25) from $\kappa = 1$ to $\kappa = k$, k being large but finite, and the integral (7.23) from $p = p_k$ to $p = 2\sqrt{\pi}$, say. Since $u_k \approx (k-\frac{1}{4})\pi$, the corresponding p_k will be $p_k = (k-\frac{1}{4})\pi/a$, and since the integrand in (7.23) varies like p^{-1} for small p, a term ln a is expected to occur.

To interpret this result, consider the amplitude equation for A_1 alone. By writing $M_{1111} = M \ln a$ where M is some positive constant, we obtain

$$(7.28) \qquad \dot{A}_1 - \sigma_1 A_1 = - A_1{}^3 M \ln a .$$

When a is large, this eigensolution (and all eigensolutions of odd order) will be strongly damped by the non linear terms, and the stationary amplitude A_1 will be small of order $(\sigma_1/\ln a)^{\frac{1}{2}}$. The temperature distribution itself, however, shows a quite strange variation across the dish. Using (7.1), (7.3), (7.4) and (7.28) we derive

$$(7.29) \qquad \begin{aligned} \theta^{(1)} &= A_1 \theta_1(\vec{r}) = \sin \pi z \left(\frac{\pi^2 \sigma_1}{M 2\sqrt{2}}\right)^{\frac{1}{2}} \left(\frac{a}{\ln a}\right)^{\frac{1}{2}} \\ &\times \left\{J_0\left(\frac{\pi r}{\sqrt{2}} + \frac{\pi r}{2a}\right) + J_0\left(\frac{\pi r}{\sqrt{2}} - \frac{\pi r}{2a}\right)\right\} . \end{aligned}$$

Near the center of the dish, $\theta^{(1)}$ is of order $(a/\ln a)^{\frac{1}{2}}$, increasing with a. Near the outer wall, it decreases with increasing a like $(\ln a)^{-\frac{1}{2}}$.

The fact that A_α is small when α is odd, has some consequences. First we note that the effect of the odd modes on the even modes will be small. Another result (to be discussed in the next section) is that a stationary solution of odd modes will be more unstable than a solution of even modes. It is therefore assumed that when a stationary axisymmetric solution is discussed, it will be sufficient to consider only the even modes.

As a first approximation the amplitude equations for A_2 and A_4 will be discussed. The equations are

$$\begin{aligned} \dot{A}_2 - \sigma_2 A_2 = -\{M_{2222}A_2^{\,3} + 3\,M_{2224}A_2^{\,2}A_4 \\ + 3\,M_{2244}A_2\,A_4^{\,2} + M_{2444}A_4^{\,3}\}, \end{aligned} \tag{7.30}$$

$$\begin{aligned} \dot{A}_4 - \sigma_4 A_4 = -\{M_{4222}A_2^{\,3} + 3M_{4422}A_2^{\,2}A_4 \\ + 3M_{4442}A_2A_4^{\,2} + M_{4444}A_4^{\,3}\}. \end{aligned} \tag{7.31}$$

We shall find it convenient to introduce new variables

$$X = \frac{\sigma_2}{M_{2222}A_2^{\,2}}\,, \qquad Y = \frac{A_4}{A_2}\,, \tag{7.32}$$

and obtain the equations in the form

$$\dot{X} = 2\sigma_2(\phi(Y) - X)\,, \tag{7.33}$$

$$X\dot{Y}/Y = \sigma_2(\phi(Y) - X) - \sigma_4(\psi(Y) - X)\,. \tag{7.34}$$

When the numerical values of the coefficients are introduced, $\phi(Y)$ and $\psi(Y)$ are

$$\phi(Y) = 1 + 0.81\,Y + 2.33\,Y^2 + 0.47\,Y^3\,, \tag{7.35}$$

$$\psi(Y) = (\sigma_2/\sigma_4)\{0.27\,Y^{-1} + 2.33 + 1.41\,Y + 1.30\,Y^2\}. \tag{7.36}$$

For $\sigma_4/\sigma_2 < 0$, there is one solution satisfying $-0.13 < Y < 0$, $0.93 < X < 1$, for $0 < \sigma_4/\sigma_2 < 1$, the solution satisfies $-0.25 < Y < -0.13$ while the value of X is virtually constant and equal to 0.93. This solution is easily seen to be a stable solution of (7.33) and (7.34). Other solutions are found to be either unstable solutions, or solutions where $|Y|$ is large compared to 1 and therefore of no interest in the present approximation.

The solutions of (7.33) and (7.34) are shown in Fig. 2 . It is interesting to note the small variation in X and Y as σ_4/σ_2 varies, and also the smallness of Y for all σ_4/σ_2. This means that the eigensolution $\alpha = 2$ is the dominating term in the non linear solution. When R is so large that σ_6 becomes positive and compared to σ_4, the amplitude A_6 must be expected to be significant. The asymptotic results of section 4 make $\sigma_6 = 0$ when $\sigma_4/\sigma_2 = 0.625$, corresponding to $\eta = 24a^{-2}$. It is to be expected that for a given value of R, more eigensolutions will be significant in the non linear solution when the radius a is increased.

8. Stability of non linear circular rolls.

The amplitude equations derived in section 6 apply to a general non stationary solution of the non linear equations. It will now be applied to solutions which are close to the stationary circular roll solution discussed in the previous section. An investigation of the growth or decay of such solutions therefore yields a stability criterion for the circular rolls. The first order temperature will be written

$$(8.1) \qquad \theta = \sum_{\alpha} (A_\alpha + a_\alpha(t))\theta_\alpha(r,\phi,z) .$$

Both A_α and $A_\alpha + a_\alpha(t)$ satisfy the amplitude equations (6.17), and $a_\alpha(t)$ is assumed to be small enough for the non linear terms

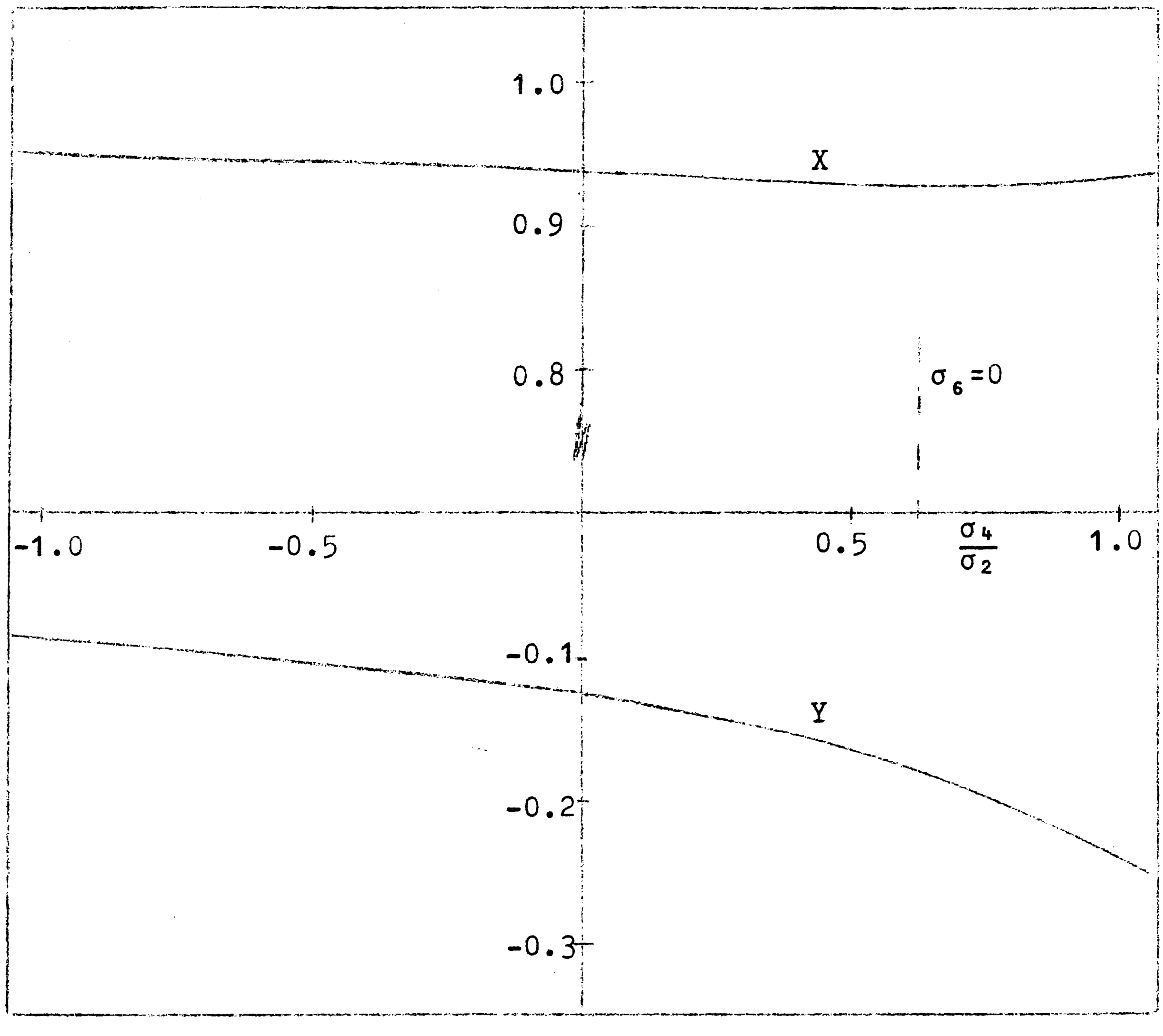

Fig. 2.

Amplitudes of stationary circular cells as functions of a and R through σ_4/σ_2.

$X = \sigma_2/(M_{2222}A_2{}^2)$, $Y = A_4/A_2$.

to be neglected. The linearized equations are

$$\dot{a}_\alpha - \sigma_\alpha a_\alpha = - \sum_{\beta\gamma\delta} a_\beta A_\gamma A_\delta (M_{\alpha\beta\gamma\delta} + M_{\alpha\gamma\beta\delta} + M_{\alpha\delta\gamma\beta}) \ . \tag{8.2}$$

It should be remembered that the indices here are index-pairs, $\alpha = (n,\alpha)$ etc. where n is the azimuthal wave number. Since A_γ and A_δ are amplitudes of an axisymmetric solution, we have $\gamma = (0,\gamma)$ $\delta = (0,\delta)$. It is then seen that the coefficients are non vanishing only when α and β both have the same azimuthal wave number. We thus write $\alpha = (n,\alpha)$ $\beta = (n,\beta)$, and consider one single value of n. We also note that the case of two solutions which are turned an angle ϕ' relative to each other need not to be considered. This is because a sum of two such solutions is itself a single solution of the same type, having a definite amplitude and orientation relative to a fixed axis. The perturbation temperature θ' is therefore written

$$\theta' = \sum_\alpha a_\alpha(t) \sin \pi z \cos n\phi \{C_{\alpha 1} J_n(q_{\alpha 1} r) + C_{\alpha 2} J_n(q_{\alpha 2} r)\}. \tag{8.3}$$

The velocity components have similar expressions, according to section 3.

When the coefficients in (8.2) are considered, it is first noted that the convective terms $\vec{u}_\alpha \cdot \nabla\theta_\gamma$ and $\vec{u}_\alpha \cdot \nabla\theta_\delta$ have their asymptotic expressions proportional to $(1 + (-1)^{\alpha+\gamma+n})$ and $(1 + (-1)^{\alpha+\delta+n})$, respectively. $M_{\alpha\gamma\beta\delta}$ and $M_{\alpha\delta\gamma\beta}$ are proportional to the same factors, and the most dangerous perturbations will be those for which these factors vanish. It is therefore sufficient to discuss the equations

$$(8.4) \qquad \dot{a}_\alpha - \sigma_\alpha a_\alpha = -\sum_{\beta\gamma\delta} a_\beta A_\gamma A_\delta M_{\alpha\beta\gamma\delta}$$

with

$$(8.5) \qquad M_{\alpha\beta\gamma\delta} = (4\pi^2 V)^{-1} \int_V (\vec{u}_\alpha \cdot \nabla\theta_\beta)(\vec{u}_\gamma \cdot \nabla\theta_\delta) dV \ .$$

Here is used the result found in section 7 that

$$(8.6) \qquad \theta_{\gamma\delta} = -\frac{1}{4\pi^2} \vec{u}_\gamma \cdot \nabla\theta_\delta \ ,$$

approximately. When the eigensolutions with wave number n are introduced in $\vec{u}_\alpha \cdot \nabla\theta_\beta$, it is found that the integral (8.5) is independent of n to the first approximation. The perturbation equations we are going to discuss are therefore (8.4) where the coefficients $M_{\alpha\beta\gamma\delta}$ have the same meaning as in the equations for the stationary solution, namely

$$(8.7) \qquad \sigma_\alpha A_\alpha = \sum_{\beta\gamma\delta} M_{\alpha\beta\gamma\delta} A_\beta A_\gamma A_\delta \ .$$

In the stationary solutions A_γ and A_δ, only even indices are of interest. Since the odd index solutions have small amplitudes of order $(\ln a)^{-\frac{1}{2}}$, they are not able to damp out a perturbation which is linearly unstable when the container is large. Also in the perturbed motion such modes can be neglected. This is because of their strong non linear damping, making the amplitude of the perturbation small (decreasing with increasing radius) at all time, even though they may grow initially.

For a perturbation consisting of the three modes (n,2),(n,4) and (n,6) the characteristic equation for the eigenvalue ω can be written

$$(8.8)\qquad \begin{vmatrix} \omega - \sigma_2 - N_2 & N_{24} & N_{26} \\ N_{42} & \omega - \sigma_4 - N_{44} & N_{46} \\ N_{62} & N_{64} & \omega - \sigma_6 - N_{66} \end{vmatrix} = 0 .$$

Here $N_{\alpha\beta}$ means

$$(8.9)\qquad N_{\alpha\beta} = \sum_{\gamma\delta} M_{\alpha\beta\gamma\delta} A_\gamma A_\delta .$$

There is always one solution $\omega_1 = 0$. When $(A_4/A_2)^2$ is small compared to one, the other solutions are approximately

$$(8.10)\qquad \omega_2 = \sigma_4 - N_{44},$$

$$(8.11)\qquad \omega_3 = \sigma_6 - N_{66}.$$

The condition for stability is $\omega_2 < 0$, which can be written

$$(8.12)\qquad \frac{\sigma_4}{\sigma_2} < \{\sum_{\gamma\delta} M_{44\gamma\delta} A_2 A_\gamma A_\delta\}\{\sum_{\beta\gamma\delta} M_{2\beta\gamma\delta} A_\beta A_\gamma A_\delta\}^{-1}.$$

With the notation of section 7, (8.12) takes the form

$$(8.13)\qquad \frac{\sigma_4}{\sigma_2} < X^{-1}(0.78 + 0.95\ Y + 1.30\ Y^2).$$

(8.13) is found to be valid for $Y > -0.18$, giving $\sigma_4/\sigma_2 < 0.70$, or

$$(8.14)\qquad \eta < 88/3a^2 .$$

The result obtained is therefore that the roll solutions discussed in section 7 are stable against perturbations with azimuthal wave numbers different from zero provided R is not too large. The value of R at which the ring cells begin to break down is estimated to be approximately

$$(8.15) \qquad R = \frac{27\pi^4}{4}(1 + 88/3a^2) .$$

In a circular container with a diameter 20 times the fluid depth, the range of Rayleigh numbers where ring cells are stable is from R_c to about $1.3\ R_c$ according to this estimate. The value of σ_4/σ_2 determined from (8.13) is found to be sensitive to changes in X and Y. The effect of A_6, which is neglected above, may therefore be significant in this criterion even though A_6 is small compared to A_2 and A_4. (8.15) should therefore be considered as a rough estimate of the stability bound.

9. Discussion and conclusion.

It has been shown in the present paper that the effect of the distant lateral walls manifests itself through the existence of a "twin solution" together with two wall layer solutions. What we call a twin solution is a sum of two roll solutions with nearly equal wave numbers and amplitudes. The wall layer solutions are solutions which decay exponentially with the distance from a vertial wall. Furthermore, the amplitudes of the wall layer solutions will decrease as the size of the container increases. In this way we have obtained asymptotic solutions which enable us to discuss the linear and non linear solutions in some details. In the linear case, a simple formula is found to relate the growth rate σ, the Rayleigh number R and the diameter D, or the channel width B. In a cylindrical container it is found that σ is decreased and R_c is increased by a term proportional to $(H/D)^2$. In a channel the correction terms are proportional to $(1 - 2\kappa^2/\pi^2)(H/B)^2$, for a given wave number κ along the channel. When $\kappa^2 - \pi^2/2$ is zero or small of order $(H/B)^2$, the correction is proportional to $(H/B)^4$. The result is therefore that the walls which are parallel to the roll axes will increase R_c by a term of order $(H/B)^2$. The increase in R_c, due to the walls perpendicular the roll axes will be of order $(H/B)^4$. This is in fact the result of Segel (1969), and it is also in accordance with the experimental fact that the rolls tend to aligne themselves parallel to the shorter side of a rectangular dish. However, these theoretical results can hardly be considered to prove the observed orientation of the rolls. Consider a rectangular dish with sides B_1 and B_2, where $B_1 > B_2$ but B_1 and B_2 are of the same order. The theory predicts the relative increase in R_c for roll solutions to be about $2.7\ (H/B_1)^2$ and $2.7\ (H/B_2)^2$ as the roll axes are parallel to the shorter or the longer sides. The difference between these values of R_c is hardly significant for a dish of the size used in the experiments referred to in the introduction. We are therefore led to the conclusion that the non linear solutions and the stability of them must be taken into account to get a criterion for the orientation of the rolls.

Such an investigation will be made in another paper. Here we just point out that the linear solutions discussed above in fact show a tendency to make rolls along the channel more unstable than rolls across the channel. Let a finite amplitude roll solution be given a perturbation of the form of rolls perpendicular to the finite rolls, and consider the growth rate σ of the perturbation. There is a tendency for σ to be smallest when the perturbation rolls are along the channel for two reasons. First we note that these rolls have the smallest σ due to linear effects. Secondly, the amplitude of the finite rolls (proportional to the square root of the linear growth rate) will be largest when the rolls are across the channel. These rolls, thus having the largest amplitudes, will therefore be most able to damp out a perturbation.

The question of the preferred wave number has also been considered. The eigensolutions of the linearized equations were found to have a well defined wave number, nearly the same for all weakly unstable solutions. It was also found that this wave number q_o has a slight increase with R, the estimate $q_o = (R/27)^{\frac{1}{4}}$ was obtained. This will be the increase in wave number both for non linear circular rolls, and for non linear straight rolls along a channel. The lateral walls which are parallel to the roll axes therefore tend to increase the number of rolls. The effect of the walls perpendicular to the roll axes may be different. This will be a question of which wave number (measured along the channel) will correspond to a stable non linear roll solution. An attempt to answer this question was made by Davis (1968). By calculating the heat transport H of a roll solution in a box, he determines that number of rolls which convect most heat at a given value of R. While M rolls are preferred at $R = R_c$, it is found that M-1 rolls give a larger value of H than M rolls do when R exceeds some supercritical value. By assuming the maximum of H to be a criterion for the preferred solution, the conclusion was drawn that the number of rolls decreases with increasing R. It must be remembered, however, that the criterion mentioned above is not proved to be generally valid.

An increase in the observed wave length of rolls for increasing R is reported by experimenters. Koschmieder (1966,1969) found this to be the case for circular rolls both for a free and a rigid upper boundary. The same tendency was observed by Krishnamurti (1970), and partly by Hoard et al (1970). In the last mentioned paper it was reported that when rolls were formed at a value of R above R_c, the observed wave number was smaller than its theoretical (critical) value. For hexagons, however, the contrary effect was observed. The discussion given by Busse & Whitehead (1971) on the problem does not apply to the question of the preferred wave length of circular rolls. There seems therefore to be no satisfactory explanation for the variation of wave number with increasing R. The conductivity of the boundaries are usually not taken into account in theoretical works, and may be responsible for a disagreement between theory and experiments. However, it was pointed out by Koschmieder (1969) that the behaviour was not altered much as the upper boundary was a good or poor conductor. The effect of the lateral walls must be said to be in contrast to the observations as far as the present theory is valid.

It was pointed out above that for a container of the size used in experiments, the correction of R_c due to the lateral walls is very small. When the diameter of a circular dish is 30 times the fluid depth, R_c is 0.3 per cent above that for an unbounded layer. However, this is not the important effect of the lateral walls. Far more significant is the result that when R is 10 per cent above R_c, not more than 5 axisymmetric eigensolutions are linearly unstable. When R is increased to 20 per cent above R_c, the number of unstable solutions is 8. All of these solutions have the same wave number which turns out to be equal to the critical one for an unbounded layer to the first approximation. If a non linear stationary solution is given an axisymmetric perturbation, this perturbation can not contain more than these 5 (or 8) eigensolutions if the perturbation itself shall be linearly unstable. A similar result is seen to hold for the solutions in a channel. The bahaviour of these solutions is in contrast to the behaviour of the solutions in an unbounded layer. In that case an infinity of wave numbers are present in the unstable solutions. Further there are important stability criteria based on the assumption that the wave numbers of the perturbation can be chosen arbitrarily close to that of the stationary solution. In addition to these different behaviours comes the particular discrepancy for the axisymmetric solution. A non linear axisymmetric solution does not exist in an unbounded layer as pointed out in the introduction.

The paper of Joseph (1971) considers problems related to those mentioned above. What he proves is that a non linear solution which is unstable when the fluid layer has an infinite diameter, may get stable when the container is taken finite. Strictly speaking, the criterion is different as the eigenvalues are simple or not. However, as Joseph points out, his stability analysis is a linearized one. The effect of the lateral walls may therefore be less important than the non linear terms neglected, when the container is large.

The criterion of stability of circular cells which was found in section 8 should be regarded as a rough estimate only. However, the main conclusion is that the circular rolls are stable against perturbations which are not axisymmetric, up to a certain super-critical Rayleigh number. This value of R is dependent on the diameter of the dish, and decreases toward R_c as the diameter increases. This is qualitatively in accordance with observations. However, the experimental works referred to above seems to be more concerned with the values of R where different cell patterns break down, affected by the fluid property variation with temperature and different boundary conditions.

At last a comment on the boundary conditions should be made. The assumption that the horizontal boundaries could be considered as free and perfectly conducting, was crucial to get the equations solved by separation of the variables. However, the result that two roll solutions with neighbouring wave numbers can be combined to give an approximate solution, will obviously apply to other boundary conditions. The dependence on the vertical coordinate will be approximately the same for the two wave numbers. It will thus be possible to discuss the combined effect of lateral walls, surface tension and finite conductivity and rigidity at the horizontal boundaries.

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